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Eigenvectors in the superintegrable model I: $\mathfrak{sl}_2$ generators

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Abstract

In order to calculate correlation functions of the chiral Potts model, one only needs to study the eigenvectors of the superintegrable model. Here we start this study by looking for eigenvectors of the transfer matrix of the periodic $\tau_2(t_q)$ model which commutes with the chiral Potts transfer matrix. We show that the degeneracy of the eigenspace of $\tau_2(t_q)$ in the $Q = 0$ sector is 2^r, with $r = (N-1)L/N$ when the size of the transfer matrix L is a multiple of N. We introduce chiral Potts model operators, different from the more commonly used generators of quantum group $\tilde{\mathrm{U}}_q(\widehat{\mathfrak{sl}}_2)$. From these we can form the generators of a loop algebra $\mathcal{L}(\mathfrak{sl}_2)$. For this algebra, we then use the roots of the Drinfeld polynomial to give new explicit expressions for the generators representing the loop algebra as the direct sum of r copies of the simple algebra $\mathfrak{sl}_2$.

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The integrable chiral Potts model is an N-state spin model on a planar lattice, whose Boltzmann weights require high-genus algebraic functions for their parameterization [1–5]. Nevertheless much progress has been made. The model has very special properties which made it possible for Baxter to calculate the free energy and order parameters [6–9]. It seems likely that correlation functions of this model can also be calculated.

Unlike the calculation of the free energy and order parameters, for which the knowledge of the eigenvalues of the transfer matrix is sufficient, to calculate the correlation functions, we also need information about the eigenvectors. We shall show that it is only necessary to study the eigenvector space in the superintegrable model, which is in many ways similar to the Ising model [10, 11]. Particularly, one can construct a loop algebra in the superintegrable model similar to the Onsager algebra [10] in the Ising model.

We consider here the square lattice drawn diagonally with edges denoted by solid lines as in figure 1. Each spin σ may take N different values, and interacts with each neighboring spin σ'. Boltzmann weights $W_{pq}(\sigma - \sigma')$ (or $\bar{W}_{pq}(\sigma - \sigma')$) are associated with pair interactions

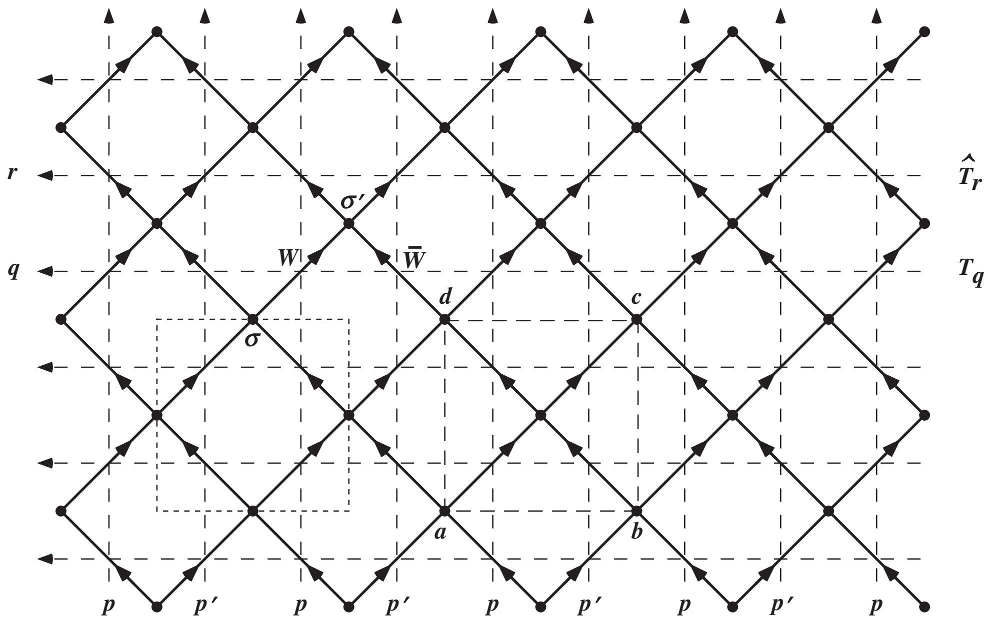

Figure 1. The oriented lattice of a two-dimensional spin model is represented by solid lines. At each lattice site, indicated by a black dot, there is a spin which can take N different values. The Boltzmann weights W and $\overline{W}$ are associated with pair interactions along the edges. The medial graph consists of oriented dashed lines—the 'rapidity lines'—carrying variables $p, q, \ldots$. To each rapidity line q (or r), we associate a transfer matrix T_q (or $\hat{T}_r$) and these transfer matrices commute with one another for a Z-invariant lattice. The product of two transfer matrices $T_q\hat{T}_r$ can be written as products of 'stars' $\mathcal{U}(a, b, c, d)$, one of which is drawn inside the dashed box; assuming periodic boundary conditions, it can also be written as a trace of the product of intertwiners $\mathcal{S}$, 'squares', shown to the left inside the dotted box.

along the SW–NE edges (or the SE–NW edges). The rapidity variables p and q are denoted by the dashed oriented lines in figure 1. The Boltzmann weights are given in product forms as

$$W_{pq}(n) = \left(\frac{\mu_p}{\mu_q}\right)^n \prod_{j=1}^{n} \frac{y_q - x_p\omega^j}{y_p - x_q\omega^j}, \qquad \overline{W}_{pq}(n) = (\mu_p\mu_q)^n \prod_{j=1}^{n} \frac{\omega x_p - x_q\omega^j}{y_q - y_p\omega^j}, \tag{1}$$

where $\omega^N = 1$, $W_{pq}(N+n) = W_{pq}(n)$ and $\overline{W}_{pq}(N+n) = \overline{W}_{pq}(n)$. To each horizontal and vertical rapidity line p, two variables x_p and y_p are assigned and they are related by the equations[1]

$$\mu_p^N = k'/\big(1 - kx_p^N\big) = \big(1 - ky_p^N\big)/k', \qquad x_p^N + y_p^N = k\big(1 + x_p^N y_p^N\big), \tag{2}$$

$k^2 + k'^2 = 1$, which determine a high-genus curve. The curve is symmetric in x_p and y_p, and there exist various automorphisms, such as $p \to Rp$, $p \to Up$,

$$(x_{Rp}, y_{Rp}, \mu_{Rp}) = (y_p, \omega x_p, 1/\mu_p), \qquad (x_{Up}, y_{Up}, \mu_{Up}) = (\omega x_p, y_p, \mu_p), \tag{3}$$

which leave the curve invariant.

[1] Korepanov introduced a version of the τ_2-model in 1986 and discovered the genus $(N-1)^2$ curve in the second member of (2) as its 'vacuum curve' using Baxter's 'pair-propagation through a vertex' method to find the condition under which the transfer matrices commute. Korepanov's work was not known to us when we discovered the genus-10 curve for the three-state chiral Potts model late in 1986 [1], as it became available in the West only 7 years later [12].

To each horizontal rapidity line q, r one associates a transfer matrix, which is the product of all the weights,

$$T_q = T(x_q, y_q)_{\sigma\sigma'} = \prod_{J=1}^{L} W_{pq}(\sigma_J - \sigma'_J)\overline{W}_{p'q}(\sigma_{J+1} - \sigma'_J),$$
$$\hat{T}_r = \hat{T}(x_r, y_r)_{\sigma'\sigma''} = \prod_{J=1}^{L} \overline{W}_{pr}(\sigma'_J - \sigma''_J)W_{p'r}(\sigma'_J - \sigma''_{J+1}). \tag{4}$$

When the rapidities of the two transfer matrices T_q and $\hat{T}_r$ are related by $x_r = y_q$ and $y_r = \omega^j x_q$, the product of these two transfer matrices decomposes [13], i.e.,

$$T_q\hat{T}_r \to \bar{h}_j \tau_j(t_q) + h_j \tau_{N-j}(\omega^j t_q), \qquad t_q = x_q y_q. \tag{5}$$

Furthermore, as the matrices $\tau_j(t_q)$ still commute with the transfer matrices, they satisfy additional equations. These functional relations [13] were used by Baxter [6, 9] to calculate the free energy and the order parameters.

It is known that $\tau_2(t_q)$ can be written as a trace of the product of $\mathcal{L}$ matrices which is defined by what is in the dotted box in figure 1,

$$\tau_2(t_q) = \mathrm{Tr}[\mathcal{L}_L(t_q) \cdots \mathcal{L}_2(t_q)\mathcal{L}_1(t_q)]. \tag{6}$$

Bazhanov and Stroganov [14] have shown that these $\mathcal{L}$ matrices satisfy the Yang–Baxter equation $\mathcal{R}\mathcal{L}\mathcal{L}' = \mathcal{L}'\mathcal{L}\mathcal{R}$, in which $\mathcal{R}$ is the six-vertex R-matrix in the vertex language. This shows that the eigenspace of $\tau_2(t_q)$ is related to the representations of the quantum group $U_q(\widehat{\mathfrak{sl}}_2)$.

Since the chiral Potts model satisfies the star-triangle equation, rapidity lines can be moved through vertices without changing the partition function Z, whence such models are called Z-invariant [15]. Consequently, the order parameters depend only on the temperature variable k, whereas a pair correlation function depends only on k and the rapidity lines sandwiched between the two spins. Because of this, the diagonal correlation function of two spins separated in the vertical direction in figure 1 depends only on the horizontal rapidity lines and is independent of p and p', which we may choose such that $x'_p = y_p$ and $y'_p = x_p$ implying that the model becomes superintegrable. On the other hand, the diagonal correlations in the horizontal direction are functions of p and p' only. Thus they are intimately related to correlation functions of $\tau_j(t_q)$. For $j = 2$, especially for certain open boundary conditions, some work has recently been done on the eigenvectors of $\tau_2(t_q)$ (see, e.g., [16–18]). Here we shall use a different approach in order to get some more understanding of the consequences of superintegrability.

For the superintegrable case, $\tau_2(t_q)$ has simple eigenvalues given by [19, 20, 22]

$$\tau_2(t_q)\nu_Q = [(1-\omega t)^L + \omega^{-Q}(1-t)^L]\nu_Q, \qquad Q \in \mathbb{Z}_N, \qquad t = t_q/t_p, \tag{7}$$

where the ν_Q are common eigenvectors for all $\tau_j(t_q)$, the product of two transfer matrices and also the spin shift operator $\mathcal{X}$, which shifts all spins by 1, i.e. $\sigma_j \to \sigma_j + 1$. The functional relations are matrix equations, but as these matrices commute, Baxter treated them as relations between the eigenvalues; that is he treated these relations as scalar equations. For the correlations, one needs information about the eigenvectors, and we must treat the functional relations as matrix identities as Tarasov did [22], that is, the matrices are expressed in terms of the eigenvectors of the spin shift operator $\mathcal{X}$.

The eigenvectors of $\mathcal{X}$ are

$$|Q; n_1, \ldots, n_L\rangle = |Q; \{n_j\}\rangle = N^{-\frac{1}{2}} \sum_{\sigma_1=0}^{N-1} \omega^{-Q\sigma_1} |\sigma_1, \sigma_2, \ldots, \sigma_L\rangle, \left(\sigma_j \equiv \sigma_1 - \sum_{k=1}^{j-1} n_k\right) \tag{8}$$

where $n_j = \sigma_j - \sigma_{j+1}$ is the difference between the adjacent spins and can be considered as variables associated with the j th edge. The cyclic boundary condition $\sigma_{L+1} = \sigma_1$ becomes $n_1 + \cdots + n_L = 0 \pmod N$. Obviously, $\mathcal{X}|Q; \{n_j\}\rangle = \omega^Q|Q; \{n_j\}\rangle$.

In the superintegrable case, the N functional relations

$$T_Q(x_q, y_q)\hat{T}_Q(y_q, \omega^j x_q)v_Q = C\mathcal{P}(t)v_Q, \qquad t = t_q/t_p \tag{9}$$

are all the same, independent of j, where C is a factor collecting the poles of the left-hand side [20]. The transfer matrix, which depends only on the difference $\ell = \sigma_1 - \sigma_1'$ in the space of the eigenvectors of the spin shift operator $\mathcal{X}$, is

$$\langle\{n_j'\}|T_Q(x_q, y_q)|\{n_j\}\rangle = \langle Q; \{n_j'\}|T_q|Q; \{n_j\}\rangle = \sum_{\ell=0}^{N-1} \omega^{-Q\ell}(T_q)_{\sigma,\sigma'}, \tag{10}$$

which is the Fourier transform over ℓ, whereas

$$\mathcal{P}(t) = P(t^N) = \frac{t^{-Q}}{N} \sum_{n=0}^{N-1} \omega^{-nQ} \frac{(1-t^N)^L}{(1-\omega^n t)^L}. \tag{11}$$

We shall consider the case $Q = 0$ only. For other cases, see [20] for details. The function $\mathcal{P}(t)$ is a polynomial in $z = t^N$ and is of order $r = (N-1)L/N$. From (9) and the r zeros of (11), Baxter obtained 2^r eigenvalues of the transfer matrix T_q, and thus found the free energy. What is implicit in Baxter's calculation is that corresponding to the $Q = 0$ eigenvalue in (7), there are 2^r eigenvectors; the eigenspace of $\tau_2(t_q)$ is highly degenerate. For $N = 3$ and small number of sites L, we can calculate all the eigenvalues and eigenvectors of $\tau_2(t_q)$ and find that the degeneracy is 2^r with $r = (N-1)L/N$ only if L is a multiple of N. These results agree with those of Deguchi [23].

Instead of (6), we let $\tau_2(t_q)$ be written as a product of $\mathcal{U}(a, b, c, d)$, which is what is inside the dashed box in figure 1,

$$\tau_2(t_q) = \prod_{J=1}^{L} \mathcal{U}(\sigma_J, \sigma_{J+1}, \sigma_{J+1}', \sigma_J'). \tag{12}$$

It is easily shown that the Yang–Baxter equation $\mathcal{R}\mathcal{U}\mathcal{U}' = \mathcal{U}'\mathcal{U}\mathcal{R}$ also holds, but now it is in the IRF language.

For the superintegrable case, the only nonvanishing elements of $\mathcal{U}$ are

$$\begin{aligned} &\mathcal{U}(a, b, b, a) = 1 - \omega^{n+1}t, \qquad &&\mathcal{U}(a, b, b-1, a) = \omega t(\omega^{n+1} - 1), \\ &\mathcal{U}(a, b, b, a-1) = (1 - \omega^n), \qquad &&\mathcal{U}(a, b, b-1, a-1) = \omega(\omega^n - t), \end{aligned} \tag{13}$$

where $n = a - b$, which is one of the edge variables. We may write

$$\mathcal{U}(a, b, c, d) = \mathfrak{u}(a-d, b-c)_{d-c,a-b}. \tag{14}$$

Using the usual convention

$$\begin{aligned} &\mathbf{Z}_{n,m} = \langle n|\mathbf{Z}|m\rangle = \omega^m \delta_{n,m}, \qquad &&\mathbf{Z}|m\rangle = \omega^m|m\rangle, \\ &\mathbf{X}_{n,m} = \langle n|\mathbf{X}|m\rangle = \delta_{n,m+1}, \qquad &&\mathbf{X}|m\rangle = |m+1\rangle, \end{aligned} \tag{15}$$

we find from (13) and (14) that

$$\begin{aligned} &\mathfrak{u}(0, 0) = (1 - \omega t\mathbf{Z}), \qquad &&\mathfrak{u}(0, 1) = -\omega t(1 - \mathbf{Z})\mathbf{X}, \\ &\mathfrak{u}(1, 0) = \mathbf{X}^{-1}(1 - \mathbf{Z}), \qquad &&\mathfrak{u}(1, 1) = \omega(\mathbf{Z} - t). \end{aligned} \tag{16}$$

Due to the way the arrows on the rapidity lines are drawn in figure 1, corresponding to the choice in the original paper [4], the matrix multiplication is either down to up and right to left, or from left to right and up to down. We use the latter choice, such that the ket vectors $|\,\rangle$ are related to the variables $\{n_j\}$ on the lower edges of the L faces. Letting

$$\mathbf{X}_j = \mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes\overset{j}{\mathbf{X}}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1},\qquad \mathbf{Z}_j = \mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes\overset{j}{\mathbf{Z}}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}, \tag{17}$$

we may define

$$(1-\omega)\mathfrak{e}_j = \mathbf{X}_j^{-1}(1-\mathbf{Z}_j),\qquad (1-\omega)\mathfrak{f}_j = (1-\mathbf{Z}_j)\mathbf{X}_j, \tag{18}$$

such that

$$(1-\omega)(\mathfrak{e}_j\mathfrak{f}_j-\omega\mathfrak{f}_j\mathfrak{e}_j) = \left(1-\omega\mathbf{Z}_j^2\right), \tag{19}$$

which relations are not the same as the ones for the cyclic and nilpotent representation of $U_q(\mathfrak{sl}_2)$ used by Jimbo [24, 25]. Nevertheless, it is easy to show that $\mathfrak{e}^N=0$ and $\mathfrak{f}^N=0$, and

$$\mathfrak{e}|0\rangle = 0,\qquad \mathfrak{e}|n\rangle = [n]|n-1\rangle,\qquad \mathfrak{f}|N-1\rangle = 0,\qquad \mathfrak{f}|n\rangle = [n+1]|n+1\rangle, \tag{20}$$

where $[n]=(1-\omega^n)/(1-\omega)=1+\omega+\cdots+\omega^{n-1}$. This definition, though different from the one more commonly used in quantum groups, is not new in the literature. This $[n]$ is also a symbol defined in q-series [27]. From (16) and (18) we may associate with each face an operator,

$$\mathfrak{u}_j = \begin{bmatrix}(1-\omega t\mathbf{Z}_j) & -\omega t(1-\omega)\mathfrak{f}_j\\ (1-\omega)\mathfrak{e}_j & \omega(\mathbf{Z}_j-t)\end{bmatrix}. \tag{21}$$

By multiplying all these operators for the L faces together, we obtain

$$\mathbf{U}(t) = \mathfrak{u}_1\cdots\mathfrak{u}_L = \begin{bmatrix}\mathbf{A}(t) & \mathbf{B}(t)\\ \mathbf{C}(t) & \mathbf{D}(t)\end{bmatrix}, \tag{22}$$

such that,

$$\tau_2(t_q)|_Q \equiv \langle Q;\{n'_j\}|\tau_2(t_q)|Q;\{n_j\}\rangle = \mathbf{A}(t)+\omega^{-Q}\mathbf{D}(t), \tag{23}$$

for $Q=0,\ldots,N-1$. From (21), we can see easily that the elements of $\mathbf{U}(t)$ are polynomials in t,

$$\mathbf{U}(t) = \sum_{j=0}^{L}(-\omega t)^j\begin{bmatrix}\mathbf{A}_j & \mathbf{B}_j\\ \mathbf{C}_j & \mathbf{D}_j\end{bmatrix}. \tag{24}$$

A few of the coefficients of these polynomials are easy to find. Particularly,

$$\mathbf{A}_0=\mathbf{D}_L=\mathbf{1},\qquad \mathbf{A}_L=\mathbf{D}_0\omega^{-L}=\prod_{j=1}^{L}\mathbf{Z}_j,\qquad \mathbf{C}_L=\mathbf{B}_0=0; \tag{25}$$

$$\mathbf{B}_L=(1-\omega)\sum_{j=1}^{L}\prod_{m=1}^{j-1}\mathbf{Z}_m\mathfrak{f}_j,\qquad \mathbf{C}_0=(1-\omega)\sum_{j=1}^{L}\omega^{j-1}\prod_{m=1}^{j-1}\mathbf{Z}_m\mathfrak{e}_j,$$
$$\mathbf{B}_1=(1-\omega)\sum_{j=1}^{L}\omega^{L-j}\mathfrak{f}_j\prod_{m=j+1}^{L}\mathbf{Z}_m,\qquad \mathbf{C}_{L-1}=(1-\omega)\sum_{j=1}^{L}\mathfrak{e}_j\prod_{m=j+1}^{L}\mathbf{Z}_m. \tag{26}$$

Each term in the operators B_1 and B_L raises only one of the n_j's to n_j+1 and the resulting state does not satisfy the cyclic boundary condition $\sum n_j = \ell N$; thus we need to consider the product of N of them. Define

$$\mathbf{B}_j^{(N)} = \frac{\mathbf{B}_j^N}{[N]!}, \qquad \text{for} \quad j = 1 \text{ or } L,$$
$$\mathbf{C}_n^{(N)} = \frac{\mathbf{C}_n^N}{[N]!}, \qquad \text{for} \quad n = 0 \text{ or } L-1, \tag{27}$$

with $[N]! = [N]\cdots[2][1]$. More generally, we define $\mathbf{O}^{(n)} = \mathbf{O}^n/[n]!$ for operator $\mathbf{O}$ and $n = 1, 2, \ldots, N$. From the Yang–Baxter equation, we find for $Q = 0$,

$$\begin{aligned}
\left[\tau_2(t_q), \mathbf{B}_L^{(N)}\right] &= (\omega-1)\mathbf{B}(t)\mathbf{B}_L^{(N-1)}(\mathbf{A}_L - 1),\\
\left[\tau_2(t_q), \mathbf{B}_1^{(N)}\right] &= (1-\omega^{-1})t^{-1}\mathbf{B}(t)\mathbf{B}_1^{(N-1)}(\mathbf{D}_0 - 1),\\
\left[\tau_2(t_q), \mathbf{C}_0^{(N)}\right] &= (\omega-1)\mathbf{C}(t)\mathbf{C}_0^{(N-1)}(\mathbf{D}_0 - 1),\\
\left[\tau_2(t_q), \mathbf{C}_{L-1}^{(N)}\right] &= (\omega-1)\omega t\mathbf{C}(t)\mathbf{C}_{L-1}^{(N-1)}(\mathbf{A}_L - 1).
\end{aligned} \tag{28}$$

Let $\mathcal{S}$ be the set of all vectors $|\psi\rangle = |n_1, \ldots, n_L\rangle$, which satisfy the cyclic boundary condition $\sum n_j = \ell N$, then we find from (25) that $\mathbf{A}_L|\psi\rangle = |\psi\rangle$; if L is a multiple of N then $\mathbf{D}_0|\psi\rangle = |\psi\rangle$. Thus in $\mathcal{S}$ the four operators (27) commute with $\tau_2(t_q)$. Denoting $|\Omega\rangle = |0, \ldots, 0\rangle$ and $|\bar{\Omega}\rangle = |N-1, \ldots, N-1\rangle$, which are the 'ferromagnetic' and 'antiferromagnetic' ground states, we find from (13) that

$$\begin{aligned}
\mathbf{A}(t)|\Omega\rangle &= (1-\omega t)^L|\Omega\rangle, &\qquad \mathbf{D}(t)|\Omega\rangle &= (1-t)^L|\Omega\rangle,\\
\mathbf{A}(t)|\bar{\Omega}\rangle &= (1-t)^L|\bar{\Omega}\rangle, &\qquad \mathbf{D}(t)|\bar{\Omega}\rangle &= (1-\omega t)^L|\bar{\Omega}\rangle.
\end{aligned} \tag{29}$$

Comparing with (7), we find that they are eigenvectors of $\tau_2(t_q)$ in this degenerate eigenspace. The commutation relations in (28) show that other eigenvectors can also be obtained by operating the raising operators on $|\Omega\rangle$, or the lowering operators on $|\bar{\Omega}\rangle$. It is also obvious that the eigenspace is more degenerate when L is a multiple of N.

We now let $L = \ell N$ and show the connection with the loop algebra $\mathcal{L}(\mathfrak{sl}_2)$ using the notation of Drinfeld,

$$\begin{aligned}
(1-\omega)^N\mathbf{x}_0^- &= \mathbf{B}_L^{(N)}, &\qquad (1-\omega)^N\mathbf{x}_1^- &= \mathbf{B}_1^{(N)},\\
(1-\omega)^N\mathbf{x}_0^+ &= \mathbf{C}_0^{(N)}, &\qquad (1-\omega)^N\mathbf{x}_{-1}^+ &= \mathbf{C}_{L-1}^{(N)}.
\end{aligned} \tag{30}$$

The generators of the loop algebra $\mathcal{L}(\mathfrak{sl}_2)$ are required to satisfy the following relations:

$$\mathbf{h}_0 = \left[\mathbf{x}_0^+, \mathbf{x}_0^-\right] = \left[\mathbf{x}_{-1}^+, \mathbf{x}_1^-\right], \qquad [\mathbf{h}_0, \mathbf{x}_i^-] = 2\mathbf{x}_i^-, \qquad \left[\mathbf{h}_0, \mathbf{x}_{-i}^+\right] = -2\mathbf{x}_{-i}^+ \tag{31}$$

$$\left[\mathbf{x}_{-i}^+, \left[\mathbf{x}_{-i}^+, \left[\mathbf{x}_{-i}^+, \mathbf{x}_j^-\right]\right]\right] = 0, \qquad \left[\mathbf{x}_i^-, \left[\mathbf{x}_i^-, \left[\mathbf{x}_i^-, \mathbf{x}_{-j}^+\right]\right]\right] = 0, \qquad i \neq j, \tag{32}$$

with $i, j = 0, 1$. From the relations (19) for the raising and lowering operator in (18), which differ from those of the quantum group, we can show that the operators in (26) do not satisfy the Serre relation. Therefore, the proof used by the authors in [28] to prove (32) for (30) cannot be repeated here. To prove these relations, we need q-series identities at root-of-unity, which are not available in the literature.

However, the Yang–Baxter equation can be used to show that (31) holds in the sector $\mathcal{S}$ in which all states satisfy the periodic boundary condition. We can also prove that the other identities hold for certain states,

$$\begin{aligned}
&\left[\mathbf{x}_0^+, \left[\mathbf{x}_0^+, \left[\mathbf{x}_0^+, \mathbf{x}_1^-\right]\right]\right](\mathbf{x}_1^-)^{(n)}|\Omega\rangle = 0, &\qquad &\left\{\left[\left[\mathbf{x}_{-1}^+, \mathbf{x}_1^-\right], \mathbf{x}_1^-\right] - 2\mathbf{x}_1^-\right\}(\mathbf{x}_1^-)^{(n)}|\Omega\rangle = 0,\\
&\left[\mathbf{x}_{-1}^+, \left[\mathbf{x}_{-1}^+, \left[\mathbf{x}_{-1}^+, \mathbf{x}_0^-\right]\right]\right](\mathbf{x}_0^-)^{(n)}|\Omega\rangle = 0, &\qquad &\left\{\left[\left[\mathbf{x}_0^+, \mathbf{x}_0^-\right], \mathbf{x}_0^-\right] - 2\mathbf{x}_0^-\right\}(\mathbf{x}_0^-)^{(n)}|\Omega\rangle = 0.
\end{aligned} \tag{33}$$

We have used Maple to check if the identities in (32) hold in $\mathcal{S}$ for small systems with $N = 3$, $L = 6$ and $N = 4$, $L = 8$. For the former case, the set $\mathcal{S}$ consists of $3^5 = 243$ states, and for all of them we have found that these identities hold. For the latter case, for which there are $4^7 = 16384$ states in $\mathcal{S}$, we have used a random number generator to pick up states randomly and to verify that the identities indeed hold. From the large number of checks that we have made, we conclude confidently that the conditions in (32) hold for the set $\mathcal{S}$.

As a consequence, the loop algebra

$$\mathbf{h}_m = \big[\mathbf{x}^+_{m-\ell}, \mathbf{x}^-_\ell\big], \qquad \mathbf{x}^\pm_{m+\ell} = \mp\tfrac{1}{2}\big[\mathbf{h}_m, \mathbf{x}^\pm_\ell\big], \qquad \ell, m \in \mathbb{Z} \tag{34}$$

can be defined on the sector $\mathcal{S}$. Furthermore, from (26), (27) and (30), and using notations introduced by Deguchi [23], we may calculate explicitly

$$\begin{aligned}
(\mathbf{x}^-_0)^{(n)} &= \sum_{\substack{\{0\leqslant\nu_m\leqslant N-1\}\\ \nu_1+\cdots+\nu_L=nN}} \prod_{m=1}^{L} \frac{\mathfrak{f}_j^{\nu_m}}{[\nu_m]!}\mathbf{Z}_m^{\sum_{\ell>m}\nu_\ell},\\
\big(\mathbf{x}^+_0\big)^{(n)} &= \sum_{\substack{\{0\leqslant\nu_m\leqslant N-1\}\\ \nu_1+\cdots+\nu_L=nN}} \prod_{m=1}^{L} \mathbf{Z}_m^{\sum_{\ell>m}\nu_\ell}\frac{\omega^{m\nu_m}\mathfrak{e}_j^{\nu_m}}{[\nu_m]!},\\
(\mathbf{x}^-_1)^{(n)} &= \sum_{\substack{\{0\leqslant\nu_m\leqslant N-1\}\\ \nu_1+\cdots+\nu_L=nN}} \prod_{m=1}^{L} \frac{\omega^{-m\nu_m}\mathfrak{f}_j^{\nu_m}}{[\nu_m]!}\mathbf{Z}_m^{\sum_{\ell<m}\nu_\ell},\\
\big(\mathbf{x}^+_{-1}\big)^{(n)} &= \sum_{\substack{\{0\leqslant\nu_m\leqslant N-1\}\\ \nu_1+\cdots+\nu_L=nN}} \prod_{m=1}^{L} \mathbf{Z}_m^{\sum_{\ell<m}\nu_\ell}\frac{\mathfrak{e}_j^{\nu_m}}{[\nu_m]!},
\end{aligned} \tag{35}$$

where the summations are over the L variables ν_m for $m = 1, \ldots, L$. These equations and (20) are used to find

$$\mathbf{h}_0|\Omega\rangle = \big(\mathbf{x}^+_{-1}\big)(\mathbf{x}^-_1)|\Omega\rangle = \big(\mathbf{x}^+_0\big)\big(\mathbf{x}^-_0\big)|\Omega\rangle = -r|\Omega\rangle, \tag{36}$$

$$\big(\mathbf{x}^+_0\big)^{(n)}(\mathbf{x}^-_1)^{(n)}|\Omega\rangle = \big(\mathbf{x}^+_{-1}\big)^{(n)}(\mathbf{x}^-_0)^{(n)}|\Omega\rangle = \Lambda_n|\Omega\rangle, \tag{37}$$

$$\mathbf{h}_0|\bar\Omega\rangle = -(\mathbf{x}^-_1)\big(\mathbf{x}^+_{-1}\big)|\bar\Omega\rangle = -(\mathbf{x}^-_0)\big(\mathbf{x}^+_0\big)|\bar\Omega\rangle = r|\bar\Omega\rangle, \tag{38}$$

$$(\mathbf{x}^-_1)^{(n)}\big(\mathbf{x}^+_0\big)^{(n)}|\bar\Omega\rangle = (\mathbf{x}^-_0)^{(n)}\big(\mathbf{x}^+_{-1}\big)^{(n)}|\bar\Omega\rangle = \Lambda_n|\bar\Omega\rangle, \tag{39}$$

where $r = (N-1)L/N$. Equation (32) and the above results differ from those in [25]. For this reason, we give some details of our calculation as

$$\Lambda_n = \sum_{\substack{\{0\leqslant\nu_m\leqslant N-1\}\\ \nu_1+\cdots+\nu_L=nN}} 1, \qquad \mathcal{Q}(t) = \prod_{m=1}^{L}\left(\sum_{\nu_m=0}^{N-1} t^{\nu_m}\right) = \frac{(1-t^N)^L}{(1-t)^L}, \tag{40}$$

where we have inserted t^{ν_m} in each of the L sums in Λ_n to arrive at $\mathcal{Q}(t)$. The condition $\nu_1 + \cdots + \nu_L = nN$ means Λ_n is the coefficient of t^{nN} in the expansion of $\mathcal{Q}(t)$. This way we find

$$\Lambda_n = \Lambda_{r-n} = \sum_{m=0}^{n}(-1)^m\binom{L}{m}\frac{(L)_{nN-mN}}{(nN-mN)!}. \tag{41}$$

Comparing (40) with (11), we find that the polynomial in the above equation is identical to the one used by Tarasov and Baxter. According to the evaluation representation on the loop algebra [23, 29], the dimension of the eigenspace generated by these operators is 2^r. This can be seen as follows: in a similar fashion as in [23], we can show by induction

$$\left(\mathbf{x}_0^+\right)^{(n-1)}\left(\mathbf{x}_1^-\right)^{(n)}|\Omega\rangle=\sum_{j=1}^{n}\mathbf{x}_j^-\Lambda_{n-j}|\Omega\rangle,\qquad \left(\mathbf{x}_1^-\right)^{(n-1)}\left(\mathbf{x}_0^+\right)^{(n)}|\bar\Omega\rangle=\sum_{j=1}^{n}\mathbf{x}_{j-1}^+\Lambda_{n-j}|\bar\Omega\rangle. \tag{42}$$

For $n>r$, its left-hand side vanishes; there are thus only r independent $\mathbf{x}_j^-$ or $\mathbf{x}_j^+$. Particularly, for $n=r+1$, we have

$$\begin{aligned}&\sum_{j=1}^{r+1}\mathbf{x}_j^-\Lambda_{r+1-j}|\Omega\rangle=\sum_{j=0}^{r}\mathbf{x}_{j+1}^-\Lambda_j|\Omega\rangle=0,\\&\sum_{j=1}^{r+1}\mathbf{x}_{j-1}^+\Lambda_{r+1-j}|\bar\Omega\rangle=\sum_{j=0}^{r}\mathbf{x}_j^+\Lambda_j|\bar\Omega\rangle=0.\end{aligned} \tag{43}$$

Even though (42) are valid only on the 'ground states', equations (43) are valid on the entire degenerate eigenspace. This can be seen easily by applying $\mathbf{x}_m^-$ on the first and $\mathbf{x}_m^+$ on the second, and since all lowering (raising) operators commute, we find these equations are valid for the entire space generated by them. Now we can use ideas presented in Davies' paper [30]. Consider the Drinfeld polynomial in (11),

$$P(z)=\sum_{n=0}^{r}\Lambda_n z^n=\prod_{j=1}^{r}(z-z_j),\qquad z=t^N. \tag{44}$$

We may define, on the set of states where equations (43) hold,

$$\mathbf{x}_j^-=\sum_{m=1}^{r}z_m^j\mathbf{E}_m^-,\qquad \mathbf{x}_j^+=\sum_{m=1}^{r}z_m^j\mathbf{E}_m^+,\qquad \mathbf{h}_j=\sum_{m=1}^{r}z_m^j\mathbf{H}_m, \tag{45}$$

where z_j are the roots of the Drinfeld polynomial. We use (34), in which the operator on the left depends only on the sum of the indices of the operators inside the commutator, to show

$$\left[\mathbf{E}_m^+,\mathbf{E}_n^-\right]=\delta_{m,n}\mathbf{H}_m,\qquad [\mathbf{H}_m,\mathbf{E}_n^-]=2\delta_{m,n}\mathbf{E}_m^-,\qquad \left[\mathbf{H}_m,\mathbf{E}_n^+\right]=-2\delta_{m,n}\mathbf{E}_m^+. \tag{46}$$

Thus, the loop algebra is decomposed into the direct sum of r copies of $\mathfrak{sl}_2$ algebras. Moreover, it is possible though nontrivial to show that $(\mathbf{E}_j^-)^2|\Omega\rangle=0$. The degeneracy of the eigenspace of $\tau_2(t_q)$, corresponding to the eigenvalue in (7) for $Q=0$, which is generated by these r sets of operators of $\mathfrak{sl}_2$, is indeed 2^r.

We have chosen our loop algebra generators different from those used in [24–26], as we did not use the $\mathcal{L}$ matrices used in [4, 14, 22, 31], one of them being shown as a 'square' in figure 1. Rather we used the dual approach using operators $\mathcal{U}(a,b,c,d)$, one of which is indicated in figure 1 by a 'star'. The operators used here in (18) and (16) are more easily seen to be lowering and raising operators.

Even though, there is ample evidence that relations (32) hold. Yet, the proof is still lacking. Obviously, operators used by us are closely related to those used by [23–26]. Perhaps, by mapping one to the other, a proof of these identities may be found. This also will provide many interesting identities of q-series at roots of unity.

There remains a great deal to be done for cases when $Q\neq 0$. We can show that

$$\begin{aligned}&[\mathbf{A}(t)+\omega^m\mathbf{D}(t)]\mathbf{B}_L^{(N-m)}\mathbf{B}_1^{(m)}=\mathbf{B}_L^{(N-m)}\mathbf{B}_1^{(m)}[\omega^m\mathbf{A}(t)+\mathbf{D}(t)]\\&\qquad+(1-\omega)\left[(\omega t)^{-1}\mathbf{B}(t)\mathbf{B}_L^{(N-m)}\mathbf{B}_1^{(m-1)}(\mathbf{D}_0-1)+\omega^m\mathbf{B}(t)\mathbf{B}_L^{(N-m-1)}\mathbf{B}_1^{(m)}(\mathbf{A}_L-1)\right],\\&[\mathbf{A}(t)+\omega^m\mathbf{D}(t)]\mathbf{C}_0^{(N-m)}\mathbf{C}_{L-1}^{(m)}=\mathbf{C}_0^{(N-m)}\mathbf{C}_{L-1}^{(m)}[\omega^m\mathbf{A}(t)+\mathbf{D}(t)]\\&\qquad+(1-\omega)\left[t\omega\mathbf{C}(t)\mathbf{C}_0^{(N-m)}\mathbf{C}_{L-1}^{(m-1)}(\mathbf{D}_0-1)+\omega^m\mathbf{C}(t)\mathbf{C}_0^{(N-m-1)}\mathbf{C}_{L-1}^{(m)}(\mathbf{A}_L-1)\right].\end{aligned} \tag{47}$$

These are related to operators that commute with $\tau_2(t_q)|_Q$ given in (23). Using (29), we find some of the eigenvectors of $\tau_2(t_q)|_Q$ for $Q \neq 0$,

$$\begin{aligned} &[\mathbf{A}(t) + \omega^Q \mathbf{D}(t)] y_Q^- |\Omega\rangle = \omega^Q \epsilon_{-Q} y_Q^- |\Omega\rangle, \qquad && y_Q^- = \mathbf{B}_L^{(N-Q)} \mathbf{B}_1^{(Q)}, \\ &[\mathbf{A}(t) + \omega^Q \mathbf{D}(t)] z_Q^- |\bar{\Omega}\rangle = \epsilon_Q z_Q^- |\bar{\Omega}\rangle, && z_Q^- = \mathbf{C}_0^{(N-Q)} \mathbf{C}_{L-1}^{(Q)}, \end{aligned} \tag{48}$$

where $\epsilon_Q = [(1-\omega t)^L + \omega^Q (1-t)^L]$. This shows that ϵ_Q and $\omega^Q \epsilon_{-Q}$ are eigenvalues of $\mathbf{A}(t) + \omega^Q \mathbf{D}(t)$. The eigenspaces for $Q \neq 0$ are clearly seen to be much different from those for $Q = 0$. For these cases, we have not yet made much progress in finding the degeneracy of their eigenspaces, nor all the eigenvectors.

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